CHAPTER 12

Diffusion Problems - Revisited

In Part II we solved a class of steady state diffusion problems for heat conduc tion equation. In this part, we will concentrate on using the same techniques and methods to solve fluid flow problems in which diffusion of momentum is dominant.

To build a sound basis for this class of fluid flow problems, we introduce the non-dimensional form of the Navier-Stokes equation. Then, we define a measure by which we can check the weight of the diffusion process in the momentum equation.

12.1. NON-DIMENSIONAL NAVIER-STOKES EQUATIONS

The governing equation for incompressible flow of Newtonian fluid is shown in Eqn.(10.9). To compare the weight of the diffusive and convective terms, we need to make these equations non-dimensional.

Let us assume that viscous diffusivity (diffusion coefficient μ), and thermal diffusivity (conductivity κ), are both constant. Then Eqn.(10.9) can be re-written as:

$$\begin{aligned}
&\text{Continuity equation} \quad \nabla \cdot \vec{\mathbf{V}} = 0 \\
&\text{Momentum equation} \quad \frac{\partial \left(\rho \vec{\mathbf{V}}\right)}{\partial t} + \nabla \cdot \left(\vec{\mathbf{V}}\vec{\mathbf{V}}\rho\right) = -\nabla p + \mu \nabla^2 \vec{\mathbf{V}} \\
&\text{Energy equation} \quad \frac{\partial T}{\partial t} + \vec{\mathbf{V}} \cdot \nabla T = \frac{1}{\rho c_p}\left[\kappa \nabla^2 T + \dot{q}\right]
\end{aligned} \tag{12.1}$$

Furthermore, let us write these equations in the non-conservation form of the Eqn.(9.54)

$$\begin{aligned}
&\nabla \cdot \vec{\mathbf{V}} = 0 \\
&\frac{\partial \vec{\mathbf{V}}}{\partial t} + \vec{\mathbf{V}} \cdot \nabla \vec{\mathbf{V}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{\mathbf{V}} \\
&\frac{\partial T}{\partial t} + \vec{\mathbf{V}} \cdot \nabla T = \frac{\kappa}{\rho c_p}\nabla^2 T + \frac{1}{\rho c_p}\dot{q}
\end{aligned} \tag{12.2}$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

The first step in the non-dimensionalization process is to decide on the characteristic or the most important independent variables, i.e., length, temperature, viscosity, etc. These characteristics are problem dependent. For example, for flow inside a circular duct, the diameter is the characteristic length and the average velocity at some section would be the characteristic velocity. For flow over a sphere the characteristic length is the outer diameter of the sphere and the characteristic velocity is the free stream velocity.

Let us assume that we have a characteristic length L_0, a characteristic velocity U_0 and a characteristic temperature T_0. From the first two characteristic variables we can find the characteristic time as

$$\text{Time} = \frac{\text{Length}}{\text{Velocity}} \Rightarrow t_0 = \frac{L_0}{U_0}$$

The number of independent variables and the number of the non-dimensional groups (See Section 9.7) are determined by the use of the dimensional analysis and the Buckingham π-theorem[9].

It is obvious that dividing any variables by its characteristic value will result in the non-dimensional form of that variable, here shown by a star superscript . In this context we may define following non-dimensional variables.

$$\begin{array}{lll} \text{non-dimensional length:} & x^* = \dfrac{x}{L_0} & \Rightarrow \quad x = x^* L_0 \\ \text{non-dimensional velocity:} & V^* = \dfrac{V}{U_0} & \Rightarrow \quad V = V^* U_0 \\ \text{non-dimensional time:} & t^* = \dfrac{t}{L_0/U_0} & \Rightarrow \quad t = t^* L_0/U_0 \\ \text{non-dimensional temperature:} & T^* = \dfrac{T}{T_0} & \Rightarrow \quad T = T^* T_0 \\ \text{partial derivative:} & \dfrac{\partial^*}{\partial x^*} = \dfrac{1}{L_0}\dfrac{\partial}{\partial x} & \Rightarrow \quad \nabla^* = \dfrac{1}{L_0}\nabla \end{array} \tag{12.3}$$

Substituting Eqn.(12.3) into the Navier-Stokes equations (Eqn.(12.2)) will non-dimensionalize them.

1. Continuity equation

$$\nabla^* \cdot \vec{\mathbf{V}}^* = 0 \tag{12.4}$$

2. Momentum equation

$$\frac{U_0 L_0}{\nu}\left[\frac{\partial \vec{\mathbf{V}}^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* \vec{\mathbf{V}}^*\right] = -\frac{1}{\frac{\mu U_0}{L_0}}\nabla^* p + \nabla^{*2}\vec{\mathbf{V}}^* \tag{12.5}$$

Here, you can convince yourself that $\dfrac{\mu U_0}{L_0}$ has the dimension of pressure, i.e., Force/Area. Hence we can define a non-dimensional pressure p_0

$$p_0 = \frac{\mu U_0}{L_0} \Rightarrow p^* = p/p_0$$

Then, Eqn.(12.5) can be written as

$$\frac{U_0 L_0}{\nu}\left[\frac{\partial \vec{\mathbf{V}}^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* \vec{\mathbf{V}}^*\right] = -\nabla^* p^* + \nabla^{*2}\vec{\mathbf{V}}^* \tag{12.6}$$

Table 9.1 on page 156, shows that the Reynolds Number Re is a dimensionless group defined by

$$Re = \frac{U_0 L_0}{\nu}$$

Then, Eqn.(12.6) can be written as

$$Re\left[\frac{\partial \vec{\mathbf{V}}^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* \vec{\mathbf{V}}^*\right] = -\nabla^* p^* + \nabla^{*2} \vec{\mathbf{V}}^* \tag{12.7}$$

or

$$\frac{\partial \vec{\mathbf{V}}^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* \vec{\mathbf{V}}^* = \frac{1}{Re}\left[-\nabla^* p^* + \nabla^{*2} \vec{\mathbf{V}}^*\right] \tag{12.8}$$

3. Energy equation

Knowing thermal diffusivity is

$$\alpha = \frac{\kappa}{\rho c_p},$$

then the energy equation can be written as

$$\frac{1}{\alpha}\left[\frac{\partial T}{\partial t} + \vec{\mathbf{V}} \cdot \nabla T\right] = \nabla^2 T + \frac{1}{\alpha \rho c_p}\dot{q} \tag{12.9}$$

Substituting the non-dimensional terms we will have

$$\frac{L_0 U_0}{\alpha}\left[\frac{\partial T^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* T^*\right] = \nabla^{*2} T^* + \frac{L_0 U_0}{\alpha \rho c_p}\dot{q}$$

Here again you can convince yourself that $\dfrac{L_0 U_0}{\alpha \rho c_p}$ has the same dimensions as $\dot{q}$ which can be defined as the characteristic rate of heat generation

$$q_0 = \frac{L_0 U_0}{\alpha \rho c_p}$$

Then we may write the energy equation as

$$\frac{L_0 U_0}{\alpha}\left[\frac{\partial T^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* T^*\right] = \nabla^{*2} T^* + \dot{q}^*$$

Table 9.1 shows that the Peclet Number Pe is a dimensionless group defined by

$$Pe = \frac{L_0 U_0}{\alpha}$$

That is, we can finally write the energy equation as

$$Pe\left[\frac{\partial T^*}{\partial t^*} + \vec{\mathbf{V}}^* \cdot \nabla^* T^*\right] = \nabla^{*2} T^* + \dot{q}^* \tag{12.10}$$

For simplicity we drop all the star superscripts and assume that all the parameters are non-dimensional, then we might write the non-dimensional Navier-Stokes equations as

$$\begin{aligned} &\nabla \cdot \vec{\mathbf{V}} = 0 \\ &\frac{\partial \vec{\mathbf{V}}}{\partial t} + \vec{\mathbf{V}} \cdot \nabla \vec{\mathbf{V}} = \frac{1}{Re}\left[-\nabla p + \nabla^2 \vec{\mathbf{V}}\right] \\ &\frac{\partial T}{\partial t} + \vec{\mathbf{V}} \cdot \nabla T = \frac{1}{Pe}\left[\nabla^2 T + \dot{q}\right] \end{aligned} \tag{12.11}$$

It is important to notice that in the momentum and energy equations, the left hand sides include the convective terms and the right hand sides include the diffusive terms. Having the inverse of Reynolds and Peclet numbers as multipliers on the right hand side shows that the higher these numbers are the smaller the diffusive parts and the more convective the equations become. The Reynolds and Peclet numbers are the measures which show the relative weight of diffusion or convection.

12.2. DIFFUSION DOMINATED FLOWS

For diffusion dominated flow, the Reynolds number in the momentum equation should be very small. That is

$$Re << 1$$

In this case, the convective part on the left hand side would be overshadowed by the viscous diffusion on the right hand side of the equation; therefore the convective term may be dropped.

$$Re \frac{\partial \vec{\mathbf{V}}}{\partial t} \approx -\nabla p + \nabla^2 \vec{\mathbf{V}} \tag{12.12}$$

Note that because the time derivative represents the transient character of the process, not related to the relative weight of convection or diffusion, one cannot drop it. These kinds of viscous diffusion flows are called low Reynolds number or creeping flows. This equation is parabolic in time. For steady state cases the equation reduces to an elliptic form, i.e., diffusion equation, which was the subject of Part II of this book.

Similar approximation can be applied to low Peclet number energy equation. If

$$Pe << 1,$$

then, the energy equation can be written as

$$Pe\frac{\partial T}{\partial t} \approx \nabla^2 T + \dot{q} \tag{12.13}$$

which is the transient heat conduction equation.

These equations can be written in dimensional form as

$$\frac{\partial\left(\rho\vec{\mathbf{V}}\right)}{\partial t} \approx -\nabla p + \nabla\cdot\left(\mu\nabla\vec{\mathbf{V}}\right) \tag{12.14}$$

$$\frac{\partial T}{\partial t} \approx \frac{1}{\rho c_p}\left[\nabla\cdot(\kappa\nabla T) + \dot{q}\right] \tag{12.15}$$

CHAPTER 13

Steady State Creeping Flow Examples

In the previous chapter it was pointed out that for steady viscous dominated flows, the Navier-Stokes equations reduce to a set of elliptic differential equations. In part II we considered this class of equations for the conduction problems. In this chapter we apply the techniques shown in part II to solve two-dimensional creeping flows.

Creeping flow approximation is useful in many areas of fluid dynamics, including situations in which

- the fluid is very viscous, that is ν is very large, like flow of molasses,
- the fluid velocity is very small,
- the characteristic length of the problem, L_0, is very small, like flow around very small objects or flow in very narrow gaps similar to what we usually get in the lubrication.

One can easily imagine many physical problems in which one or many of these conditions are true.

13.1. TWO-DIMENSIONAL VORTICITY-STREAM FUNCTION FORMULATION

The momentum equation, in its vorticity-stream function form, was derived in Chapter11. The steady state form of this equation, as well as the equations for the stream function and the relation between stream function and vorticity are written as

$$\nabla^2\omega = 0 \tag{13.1}$$

$$\nabla^2\Psi = -\omega \tag{13.2}$$

$$u = \frac{\partial\Psi}{\partial y}, \quad \text{and} \quad v = -\frac{\partial\Psi}{\partial x} \tag{13.3}$$

Mathematically, Eqn.(13.2) is a simple diffusion equation similar to the heat conduction equation. In this equation instead of heat, momentum is diffused.

The solution strategy to solve this system of equations is the same as the one explained in Chapter 11 as

1. Solve Eqn.(13.1) for the vorticity ω,
2. Solve Eqn.(13.3) for the stream function ψ,
3. Solve Eqns.(11.1) and (11.2) for the velocities u and v

The integral form of Eqn.(13.1) is exactly similar to the case of heat conduction equation. That is, similar to Eqn.(7.24 on page 66), we can write

$$A_e\left(\frac{\partial\omega}{\partial x}\right)_e - A_w\left(\frac{\partial\omega}{\partial x}\right)_w + A_n\left(\frac{\partial\omega}{\partial y}\right)_n - A_s\left(\frac{\partial\omega}{\partial y}\right)_s + \bar{S}\Delta\forall = 0 \tag{13.4}$$

This equation states that for any control volume the net diffusion of momentum (in the form of vorticity) across the faces plus the amount of momentum generated or destructed inside that control volume should be zero.

Using a linear profile to approximate the derivatives of ω at w, e, n and s, we will have

$$\begin{aligned}
A_w\left(\frac{\partial\omega}{\partial x}\right)_w &= A_w\frac{(\omega_P - \omega_W)}{\delta x_w}\\
A_e\left(\frac{\partial\omega}{\partial x}\right)_e &= A_e\frac{(\omega_E - \omega_P)}{\delta x_e}\\
A_n\left(\frac{\partial\omega}{\partial y}\right)_n &= A_n\frac{(\omega_N - \omega_P)}{\delta y_n}\\
A_s\left(\frac{\partial\omega}{\partial y}\right)_s &= A_s\frac{(\omega_P - \omega_S)}{\delta y_s}
\end{aligned} \tag{13.5}$$

Then, similar to Eqn.(7.25), we can write

$$a_P \omega_P = \sum_{nb} a_{nb} \omega_{nb} + b \tag{13.6}$$

where

$$\begin{aligned}
a_E &= \frac{\Delta y}{\Delta x_e} \\
a_W &= \frac{\Delta y}{\Delta x_w} \\
a_N &= \frac{\Delta x}{\Delta y_n} \\
a_S &= \frac{\Delta x}{\Delta y_s} \\
b &= S_C \Delta x \Delta y \\
a_P &= \sum_{nb} a_{nb} - S_P \Delta x \Delta y
\end{aligned} \tag{13.7}$$

13.2. TWO-DIMENSIONAL LID-DRIVEN CAVITY CREEPING FLOW

Example 13.1 Two-Dimensional Lid-Driven Cavity Creeping Flow

Consider a two-dimensional square cavity filled with an incompressible fluid as shown in Figure 13.1. A steady creeping fluid motion is generated inside the cavity by the slid of the infinitely long top lid at a constant velocity U_0. Since there is no fluid squeezed out of the cavity below the moving plate, the momentum, or vorticity, generated at the upper wall is diffused into the fluid forming closed path patterns within the cavity. We want to find this pattern.

From the hydrodynamics point of view, this problem represents a simplified model of complicated flow phenomena like recirculating flows in the lubrication process or flow in micro-structures.

Working with non-dimensional equations, we may assume the size of the cavity is 1×1 and the sliding velocity is $U_0 = 1$

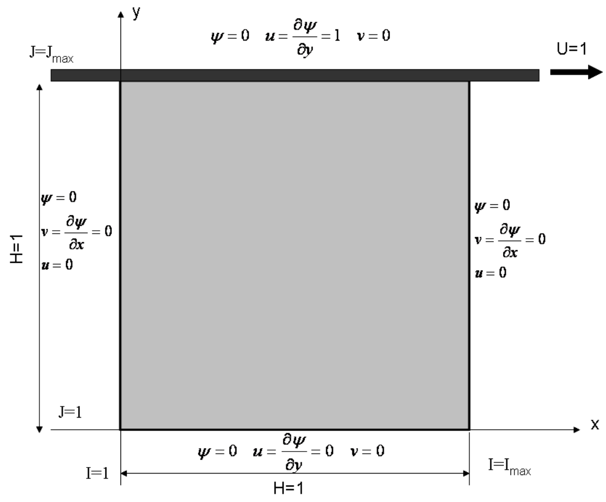

Figure 13.1: Two Dimensional Lid-Driven Cavity Creeping Flow

13.2.1. Simplified Staggered Grid

The regular approach of single grid will create some difficulties in these problems. Here, vorticity and velocities are given in terms of the derivatives of stream function. With a regular grid for vorticity, stream function and velocities on the faces of the control volumes should be approximated. Such an approximation will increase the errors.

A convenient way to get around these complications is to use two different grids. The first grid is our regular grid. We use this grid for the vorticity. The second grid is staggered towards left to coincide with the boundaries of the control volumes. This grid is used for the stream function. This is a simple model of the staggered grids which will be used later in solving convective-diffusive flows.

On this basis, we find the vorticity by solving a conservation of momentum. Then, the stream function can be found by solving the Poisson Equation, using a finite difference method. Notice that the values of the stream function are set on the second grid. Finally, having the stream function, we can find the velocities from Eqn.(13.2).

To distinguish between the two grids, the grid for vorticity is indexed by i and j and the second grid by I and J. The west walls of the cavity will be defined by $I = 1$ and $J = 1$. Assuming a uniform grid with $\Delta x = \Delta y = h$, the grids can be presented as in Figure 13.2.

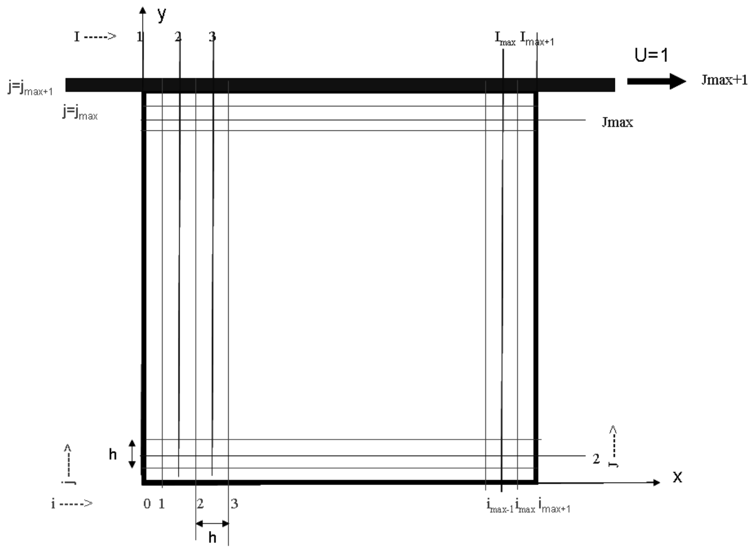

Figure 13.2: Lid-Driven Cavity Grids

13.2.2. The Integral Equations

The governing equations are shown in Eqns.(13.1) and (13.2). The vorticity equation (Eqn.(13.1)) is exactly like the heat conduction equation. This is an elliptic differential equation and the value of the vorticity should be preassigned on all boundaries. Hence, the integral equation given in the Eqn.(7.25) can be used here.

13.2.3. Boundary condition

Assigning the vorticity on the walls is not always straight forward. Here we find the vorticity on the walls by the use of Eqn.(13.3) as follows

1. The north wall

   A schematic drawing of a typical north boundary grids is shown in Figure 13.3.

   The vorticity on the north wall is given by

$$\omega_{i,j_{max+1}} = -\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)_{i,j_{max+1}}$$

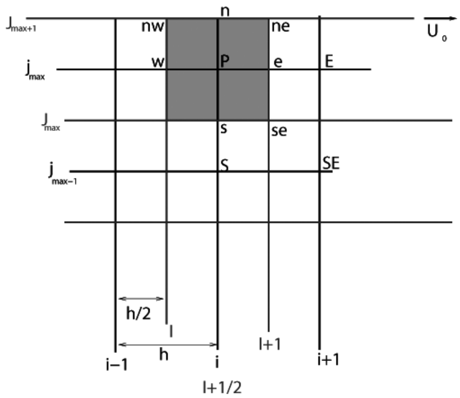

Figure 13.3: Cavity Grids-North

It is clear, from Figure 13.3, that the north control volumes are indexed by $(i, j_{max}+1)$ and the north wall by $(I+1/2, J_{max}+1)$. Then we can write

$$\omega_n = -\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)_{I+1/2, J_{max+1}}$$

On the wall, we can assume

$$\begin{aligned}\omega_n =& \frac{1}{2}\left(\omega_{nw} + \omega_{ne}\right) = \\ & -\frac{1}{2}\left[\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)_{I, J_{max+1}} + \left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)_{I+1, J_{max+1}}\right]\end{aligned} \tag{13.8}$$

First, let us consider the vorticity at $(I, J_{max}+1)$, we have

$$\omega_{nw} = -\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right)_{I, J_{max+1}}$$

It is not difficult to show that if we use the Taylor's series expansion for the right-hand side terms, we can get a proper difference equation:[10]:

$$\begin{aligned}\omega_{nw} = \frac{1}{h^2}\left(-\Psi_{I-1,J_{max}} + \frac{8}{3}\Psi_{I,J_{max}} - \Psi_{I+1,J_{max}} - \frac{2}{3}\Psi_{I,J_{max}-1}\right) \\ - \frac{2}{3h}\left(\frac{\partial\Psi}{\partial y}\right)_{I,J_{max}+1}\end{aligned} \tag{13.9}$$

Knowing

$$\left(\frac{\partial \Psi}{\partial y}\right)_{I,J_{max}+1} = u_{lid} = U_0 = 1,$$

Eqn.(13.9) can be written as

$$\omega_{nw} = \frac{1}{h^2}\left(-\Psi_{I-1,J_{max}} + \frac{8}{3}\Psi_{I,J_{max}} - \Psi_{I+1,J_{max}} - \frac{2}{3}\Psi_{I,J_{max}-1}\right) - \frac{2}{3h} = \Gamma_1 \tag{13.10}$$

Similarly we can write:

$$\omega_{ne} = \frac{1}{h^2}\left(-\Psi_{I,J_{max}} + \frac{8}{3}\Psi_{I+1,J_{max}} - \Psi_{I+2,J_{max}} - \frac{2}{3}\Psi_{I+1,J_{max}-1}\right) - \frac{2}{3h} = \Gamma_2 \tag{13.11}$$

Then, Eqn.(13.8) can be written as

$$\omega_n = \frac{1}{2}\left[\Gamma_1 + \Gamma_2\right] = \Gamma_n \tag{13.12}$$

2. The west wall

   The grids on the west wall is shown in Figure 13.4.

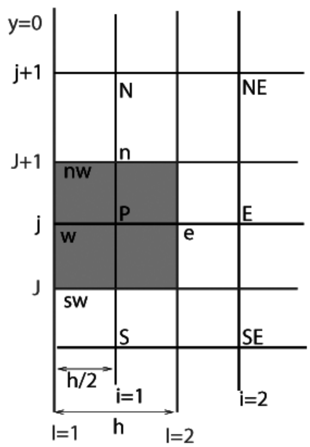

Figure 13.4: Cavity Grids-West

Here we are looking for

$$\omega_w = \frac{1}{2}\left(\omega_{nw} + \omega_{sw}\right)$$

Using the same method, we can write

$$\omega_{sw} = \frac{1}{h^2}\left(-\Psi_{2,J-1} + \frac{8}{3}\Psi_{2,J} - \Psi_{2,J+1} - \frac{2}{3}\Psi_{3,J}\right) = \Gamma_3 \tag{13.13}$$

and

$$\omega_{nw} = \frac{1}{h^2}\left(-\Psi_{2,J} + \frac{8}{3}\Psi_{2,J+1} - \Psi_{2,J+2} - \frac{2}{3}\Psi_{3,J+1}\right) = \Gamma_4 \tag{13.14}$$

Then we have

$$\omega_w = \frac{1}{2}\left[\Gamma_3 + \Gamma_4\right] = \Gamma_w \tag{13.15}$$

3. The east wall

The grids on the east wall are shown in Figure 13.5.

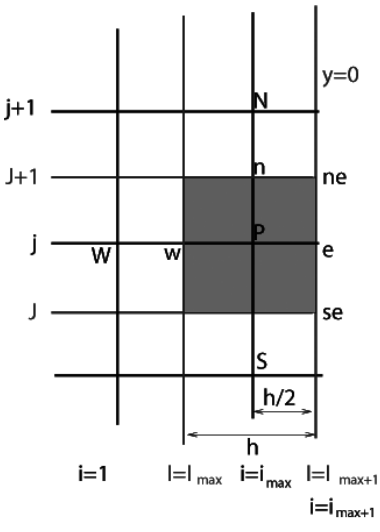

Figure 13.5: Cavity Grids-East

Here, we need to find

$$\omega_e = \frac{1}{2}\left(\omega_{se} + \omega_{ne}\right)$$